The Dark Math Under Your Car

or, Angular Velocity Ratios in Cardan Joint Systems

My fascination with u-joint math traces back to my Jeep suspension replacement project. Rear pinion angle is a critical suspension parameters and, for my jeep, the rear differential pinion shaft should be parallel to the transfer case output shaft, but different style drive shafts have different pinion angle specs, and I wanted to understand why.

A note: the pinion angle specification for my front pinion is more complicated because we have to also consider steering caster. That said, we press on with the story, but remember that what applies to the rear is modified a bit at the front.

To turn my back tires, my Jeep transfers power from the engine to the transmission and from the transmission to the transfer case in a straight line - there is no change in the axis of rotation (or, more formally, the angular momentum vector). However…

There is an angle between the transfer case shaft and the rear drive shaft (that I will call the propeller shaft), and there is another angle where the propeller shaft is connected to the rear differential pinion shaft. So the axis of rotation changes twice between the transfer case and the rear differential. If you've ever looked under your car, you probably see this: the transfer case shaft is very nearly horizontal to the ground and so is the pinion shaft, but the transfer case is higher than the pinion, so the propeller shaft must deviate from horizontal to drop down and meet the differential.

A few notes about automobile driveline configurations:

- *If you have a four wheel drive, then you should see all the components I just described carrying power to your rear tires.*
- *If you have a rear wheel, two wheel drive you won't have a transfer case and the propeller shaft will connnect directly to the transmission shaft.*
- *If you have a front wheel drive, it will be hard to see the driveline components and the arrangement is quite different, but you will still have rotational plane changes as power goes from engine down to the front wheels.*
- *Some four wheel drives use a different axle in front than in back, so the pinion specs and shaft behavior vary accordingly.*

The key to making all this work is a Cardan joint - a u-joint in automotive parlance. Extremely common, but not well understood I found out. As I researched the proper suspension set up for my Jeep, I got interested enough to work out all the math.

In this article I'll walk through just that: the dark math under the car.

I hope you enjoy it!

Cardan Joints

What is a cardan joint? Well, Girolamo Cardano was a 16th century Italian Renaissance polymath who contributed to algebra, probability, and apparently introduced the concept of imaginary numbers. In his spare time, he designed the first published Cardan joint as a mechanism to keep ship compasses steady at sea. The concept evolved and has been used since the 17th century to transfer rotary power across shafts at an angle. So, the Cardan joint is the 500 year old technology that makes your car go (kindof).

Figure 1 is a picture of a u-joint just like the ones you can buy at any autoparts store. Shaped like a cross with trunnions at each end, two rotating shafts connect to the trunnions and, when fully assembled, it is a Cardan joint that transfers rotary power across shafts at an angle.

Figure 1: *Picture of a typical u-joint that you can buy at any autoparts store. Shaft yoke ears connect to the 4 trunnions - one yoke horizontal and the other vertical*

The yoke of each shaft rotates around the axis of that shaft and in a plane normal (perpendicular) to the shaft axis. Figures 2 and 3 depict the Cardan joint in its two key configurations: In Figure 2 , the yokes lie in the plane normal to the left axis and in Figure 3, after 90° of rotation, the yokes lie in the plane of the right (angled) axis. Figure 4 is a cartoon showing how the rotational planes intersect each other. The angle between the rotational planes is the same as the angle of rotation across the joint, which I will call α.

If the descriptions and picture below leave you still unclear on how this all connects, I invite you to have a look under your car to see the arrangement in real life.

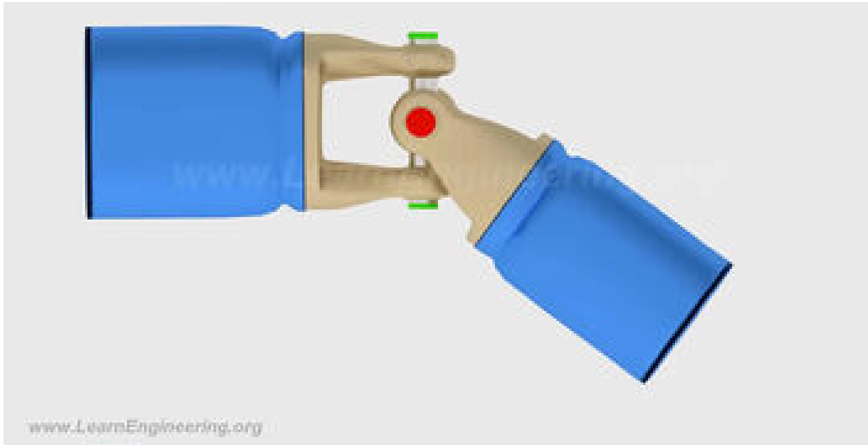

Figure 2: *Diagram of a Cardan joint. The yoke ears connect to the trunnions and, in this figure, the yokes all lie in the plane normal to the left axis. This is the starting configuration we will use as we develop math in the next sections*

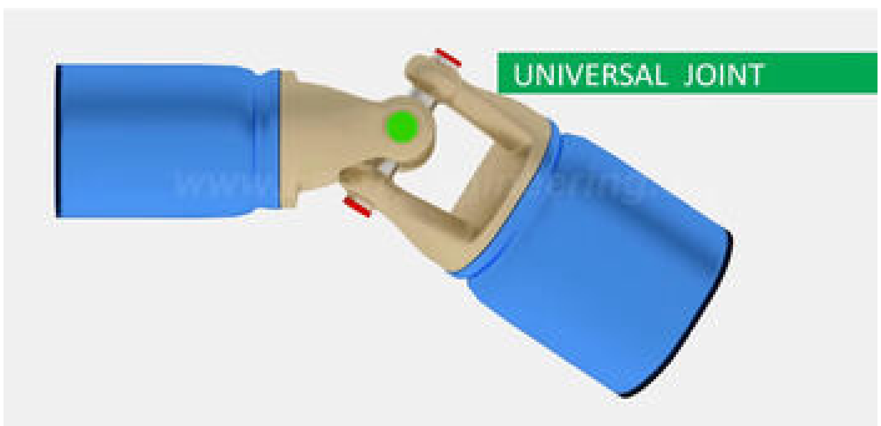

Figure 3: *Diagram of a Cardan joint. The yokes now all lie in the plane normal to the right axis. We can use either configuration to start, as long as we stay consistent*

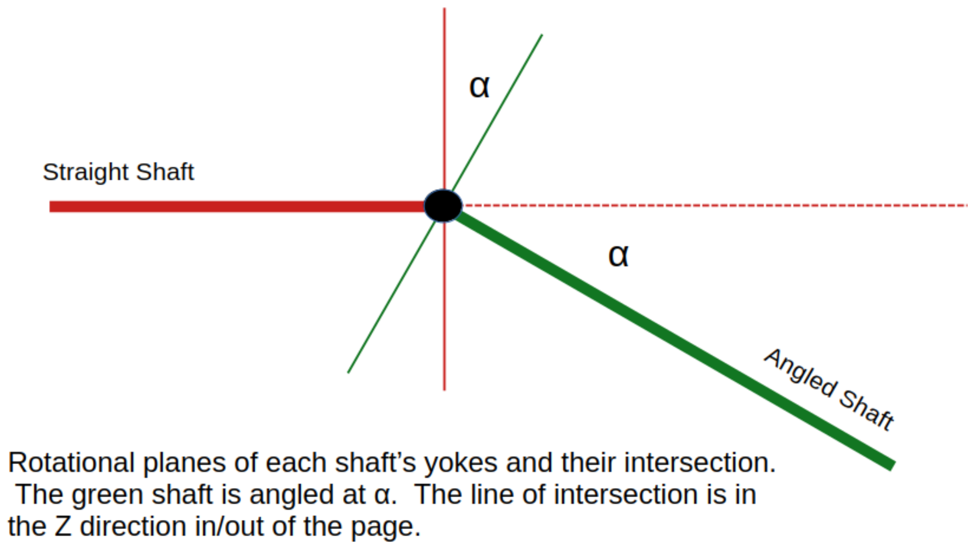

Figure 4: *Sketch showing how the rotational planes of a cardan joint intersect. The angle between planes is the same as the angle of rotation across the shafts*

I think there's no substitute for seeing this vide of a Cardan joint as it rotatwes and the narrator does a great job. So take a few minute to follow the link below before moving on to our math problem,

Follow this link to see a Cardan joint in action.

Setting the Stage

The components of my three shaft system that carries power from the transfer case to differential:

- **Shaft 1**: The transfer case shaft and the power source, rotates at angular velocity $\omega_1(\theta_1)$, and its yoke rotates in plane 1.
- **Shaft 2**: The propeller shaft transfers power from the transfer case to the differential and rotates at angular velocity $\omega_2(\theta_2)$, and its yoke rotates in plane 2.
- **Shaft 3**: The differential pinion shaft receives power and rotates at angular velocity $\omega_3(\theta_3)$, and its yoke rotates in plane 3.
- **First Cardan Joint** Shaft 1 and Shaft 2 connect at an angle α_1.
- **Second Cardan Joint** Shaft 2 and Shaft 3 at angle α_2.
- **Parallel Propeller Shaft Yokes** We will require the yokes of the propeller shaft to be parallel - this will turn out to be critical. (You can look under your car to verify that your shaft is machined accordingly.)

I am writing the instantaneous angular velocity of each shaft as a function of its rotation angle, θ, where θ ranges from 0 to 360° (or 0 to 2π).

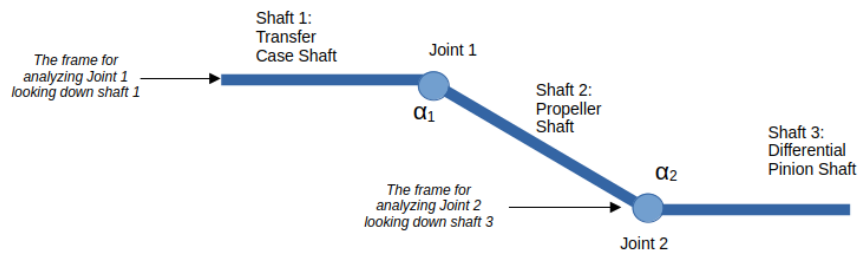

Figure 5: *Sketch showing the 3-shaft system. Shaft 1 and shaft 3 are parallel when* $\alpha_1 = -\alpha_2$

Before jumping headlong into math, lets describe in words what is happening. At the first joint, the first shaft delivers power with a constant angular velocity, ω_1. As the joint rotates, yokes rock back and forth at the trunnions, and that allow each yokes to rotate in it's own plane. It isn't obvious, but the second shaft angular velocity is not constant! At each rotation, it speeds up and then slows down, and then speeds up and slows down again, and the cycles repeat at every rotation. The average velocity of the two shafts are equal, but the instantaneous ratio, $\frac{\omega_1}{\omega_2}$ is not. Now, the trick is to set the suspension so that the second joint converts cyclic velocity back to constant velocity. Ultimately, we want the transfer case shaft and the differential pinion shaft to rotate at the same constant velocity even though the propeller shaft velocity cycles.

Sound crazy? Read on to learn the dark math.

Clocking is The Crux

Figure 6, illustrates the rotational planes of shaft 1 and 2 across the first Cardan joint looking down the axis of shaft 1. The rotational plane of shaft 1 appears as a circle. The rotational plane of shaft 2 appears as an elipse because it is rotated down at an angle α - the top of the elipse has rotated into the page away from you, and the bottom has rotated out of the page towards you.

Our starting configuration will be that of Figure 2 where both yokes lie in plane 1 - Shaft 1 yoke along our vertical y-axis and Shaft 2 yoke along our horizontal x-axis (the starting configuration isn't important as long as we stay consistent). We assume the rotation is counter clock-wise so, as it rotates, the ears of shaft 2 will move out of plane 1 - the right ear will move into the page and the left ear out of the page (we can rotate either way as long as we stay consistent).

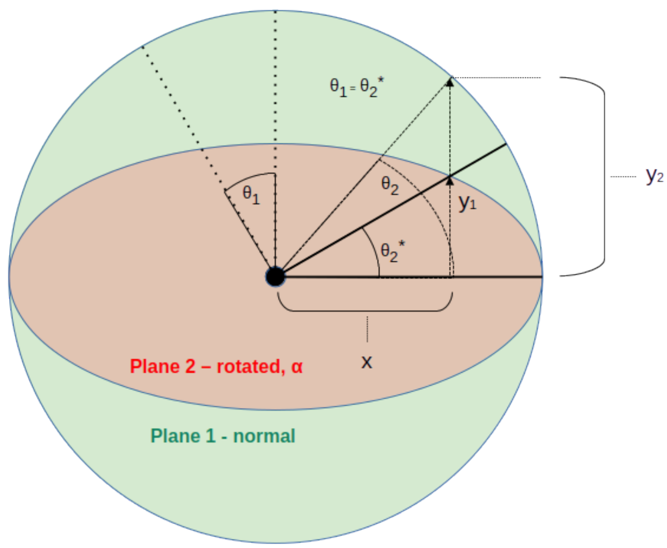

Figure 6: *View of the two rotational planes at joint 1 looking down the axis of shaft 1. Plane 2 is rotated down an angle α. When shaft 1 has rotated an angle θ_1, the clock angle of shaft 2 is θ_2^* and the actual rotation of shaft 2 is θ_2.*

We want to find the angle of rotation Shaft 1 imposes on Shaft 2 - mathematically we want to find θ_2 as a function of θ_1. The key is recognizing that the yokes of the two shafts are clocked - always 90° to each other when viewed down either Shaft axis. The angular rotation of Shaft 1 measured by its yoke is θ_1. The clock angle of shaft 2 (its apparent position when viewed down the Shaft 1 axis), is θ_2^*, which is equal to θ_1. But the shaft 2 yoke has also moved in the z direction. To find the actual rotation of Shaft 2, we mentally rotate it back into into Plane 1, as is illustrated in Figure 6, and we discover that the actual rotational angle of shaft 2 is greater than the clock angle, so: $\theta_2 > \theta_2^*$ and $\theta_2 > \theta_1$ - the propeller shaft has rotated more than the transmission shaft!

Now we must tackle some trigonometry. Recognizing from our diagram that

$$\tan(\theta_2^*) = y_1/x$$

$$\tan(\theta_2^*) = \tan(\theta_1)$$

$$\tan(\theta_2) = y_2/x$$

And

$$y_1/y_2 = \cos(\alpha)$$

So

$$\frac{\tan(\theta_2)}{\tan(\theta_1)} = \frac{y_2}{y_1} = \frac{1}{\cos(\alpha_1)}$$

Therefore

$$\boxed{\tan(\theta_2) = \tan(\theta_1)/\cos(\alpha_1)}$$

which is the relationship we want for Cardan joint 1.

Moving to the second joint, we will set α_2 equal and opposite of α_1, which makes the differential pinion shaft parallel to the transfer case shaft, which was the specification for my Jeep. We view the second joint from the same frame as the first, and we are looking down the axis of shaft 3. But the planes have swapped - the ellipse is now the rotational plane of the front shaft (shaft 2) and the circle is now the rotatoinal plane of the rear shaft (shaft 3).

To get θ_3 as a function of θ_2, we just swap the relationships we used for joint 1, and I hope the reader can easily see that:

$$\frac{\tan(\theta_3)}{\tan(\theta_2)} = \frac{y_1}{y_2} = \cos(\alpha_2)$$

$$\boxed{\tan(\theta_3) = \tan(\theta_2) * \cos(\alpha_1)}$$

It might not be obvious, but at this moment we have essentially solved our problem using the insight that the yoke ears are clocked when viewed down the axis of either shaft. To find the angular velocity relationships, we still need to take some derivatives and then transit the Sea of Trigonometry

The Sea of Trigonometry

We first find the angular position relationship for the first joint:

$$\tan(\theta_2) = \frac{\tan(\theta_1)}{\cos(\alpha_1)}$$

To find the velocity ratio ω_2/ω_1 we differentiate both sides with respect to time:

$$\frac{d}{dt}\left[\tan(\theta_2)\right] = \frac{d}{dt}\left[\frac{\tan(\theta_1)}{\cos(\alpha_1)}\right]$$

Using the chain rule and noting that α_1 is constant:

$$\sec^2(\theta_2)\frac{d\theta_2}{dt} = \frac{1}{\cos(\alpha_1)}\sec^2(\theta_1)\frac{d\theta_1}{dt}$$

Recognizing that $\frac{d\theta}{dt} = \omega$

$$\sec^2(\theta_2)\cdot\omega_2 = \frac{\sec^2(\theta_1)}{\cos(\alpha_1)}\cdot\omega_1$$

Solving for the velocity ratio:

$$\frac{\omega_2}{\omega_1} = \frac{\sec^2(\theta_1)}{\cos(\alpha_1)\cdot\sec^2(\theta_2)}$$

Now we need to express $\sec^2(\theta_2)$ in terms of θ_1 and α_1. From the original position equation:

$$\tan(\theta_2) = \frac{\tan(\theta_1)}{\cos(\alpha_1)}$$

Using the identity $\sec^2(\theta) = 1 + \tan^2(\theta)$

$$\sec^2(\theta_2) = 1 + \tan^2(\theta_2) = 1 + \frac{\tan^2(\theta_1)}{\cos^2(\alpha_1)}$$

Finding a common denominator:

$$\sec^2(\theta_2) = \frac{\cos^2(\alpha_1) + \tan^2(\theta_1)}{\cos^2(\alpha_1)}$$

Now substitute this back into our velocity ratio:

$$\frac{\omega_2}{\omega_1} = \frac{\sec^2(\theta_1)}{\cos(\alpha_1)} * \frac{\cos^2(\alpha_1)}{\cos^2(\alpha_1) + \tan^2(\theta_1)}$$

So

$$\frac{\omega_2}{\omega_1} = \frac{\sec^2(\theta_1)\cos(\alpha_1)}{\cos^2(\alpha_1) + \tan^2(\theta_1)}$$

Now, let $\tan^2(\theta_1) = \sec^2(\theta_1) - 1$:

$$\frac{\omega_2}{\omega_1} = \frac{\sec^2(\theta_1)\cos(\alpha_1)}{\cos^2(\alpha_1) + \sec^2(\theta_1) - 1}$$

Multiply numerator and denominator by $\cos^2\theta_1$

$$\frac{\omega_2}{\omega_1} = \frac{\cos^2(\theta_1)\sec^2(\theta_1)\cos(\alpha_1)}{\cos^2(\theta_1)\cos^2(\alpha_1) + \cos^2(\theta_1)\sec^2(\theta_1) - \cos^2(\theta_1)}$$

but $\sec^2(\theta_1) = \frac{1}{\cos^2(\theta_1)}$:

$$\frac{\omega_2}{\omega_1} = \frac{\cos(\alpha_1)}{\cos^2(\theta_1)\cos^2(\alpha_1) + 1 - \cos^2(\theta_1)}$$

Factor out $\cos^2(\theta_1)$:

$$\frac{\omega_2}{\omega_1} = \frac{\cos(\alpha_1)}{\cos^2(\theta_1)(\cos^2(\alpha_1) - 1) + 1}$$

And $\cos^2(\alpha) - 1 = -\sin^2(\alpha)$:

$$\frac{\omega_2}{\omega_1} = \frac{\cos(\alpha_1)}{-\cos^2(\theta_1)\sin^2(\alpha_1) + 1}$$

And the final form:

$$\boxed{\frac{\omega_2}{\omega_1} = \frac{-\cos(\alpha_1)}{\cos^2(\theta_1)\sin^2(\alpha_1) - 1}}$$

We can see that when $\theta_1 = 0 \; or \; \pi$, $\cos^2(\theta_1) = 1$ and

$$\frac{\omega_2}{\omega_1} = \frac{-\cos(\alpha_1)}{\sin^2(\alpha_1) - 1} = \frac{-\cos(\alpha_1)}{-\cos^2(\alpha_1)} = \frac{1}{\cos(\alpha_1)}$$

The propeller shaft is rotating faster than the transfer case shaft.

And when $\theta_1 = \frac{\pi}{2}$, $\frac{3\pi}{2}$, $\cos^2(\theta_1) = 0$ and

$$\frac{\omega_2}{\omega_1} = \frac{-\cos(\alpha_1)}{-1} = \cos(\alpha_1)$$

The propeller shaft is rotating slower than the transfer case shaft.

And let's not miss the observation that the differences increase as the angle of rotation, α, increases.

We still need the relationship $\frac{\omega_3}{\omega_2}$, so let's recall the rotation angle relationship for Shaft 2 and 3 from above.

$$\tan(\theta_3) = \tan(\theta_2) * \cos(\alpha_1)$$

After a similar trigonometric crossing (which I leave as a pedagogical exercise for the reader) we arrive at:

$$\boxed{\frac{\omega_3}{\omega_2} = \frac{-\cos(\alpha_1)}{\sin^2(\theta_2)\sin^2(\alpha_1) - 1}}$$

and we find that $\cos(\theta)$ has been replaced with $\sin(\theta)$, so $\frac{\omega_3}{\omega_2}$ is now 90° out of phase with $\frac{\omega_2}{\omega_1}$

The astute reader, who will have experience some unease upon learning about cyclic behavior of the propeller shaft, can now feel hopeful that the cyclic velocity will cancel across the second cardan joint. And indeed it does - providing $\alpha_1 = -\alpha_2$ - as will be demonstrates.

Pinion Shaft Velocity and why Propeller Phasing Matters

If we have equal and opposite angles at the Cardan joints so that $\alpha_1 = -\alpha_2$ and we keep the yoke ears of the propeller shaft parallel, then the angular velocity transformation at the second joint should be the inverse of the transformation at the first joint and $\frac{\omega_3}{\omega_1} = 1$.

We can prove this mathematically by using the rotational angle relatoinships found earlier.

At the first joint:

$$\tan(\theta_2) = \frac{\tan(\theta_1)}{\cos(\alpha)}$$

and at the second joint:

$$\tan(\theta_3) = \tan(\theta_2)\cos(-\alpha) = \tan(\theta_2)\cos(\alpha)$$

Substituting the first into the second:

$$\tan(\theta_3) = \frac{\tan(\theta_1)}{\cos(\alpha)} \cdot \cos(\alpha) = \tan(\theta_1)$$

And because the starting positions of yoke 1 and 3 are the same:

$$\theta_3 = \theta_1$$

And if the angular positions are always equal, then by necessity:

$$\boxed{\frac{\omega_3}{\omega_1} = 1}$$

If we set the propeller shaft yokes perpendicular, the starting positions of yoke 1 and 3 would be 90° out of phase so that:

$$\tan(\theta_3) = \frac{\tan(\theta_2)}{\cos(\alpha)}$$

and

$$\boxed{\tan}(\theta_3) = \frac{\tan(\theta_1)}{\cos^2(\alpha)}$$

And the differential pinion shaft oscillates more violently than the propeller shaft - a condition we should be desparate to avoid.

A Graph is Worth a Thousand Words

It is helpful to plot the angular velocity ratios as a function of angular position. (the Python code for making these, and other plots, is appended to the end of this article)

Figure 7 plots the velocity ratios acros the the first Cardan joint, $\frac{\omega_2}{\omega_1}$ as a function of θ_1 for several values of α. Figure 8 shows the velocity ratio at both joints - $\frac{\omega_2}{\omega_1}$ (solid lines) and $\frac{\omega_3}{\omega_2}$ (dashed lined),

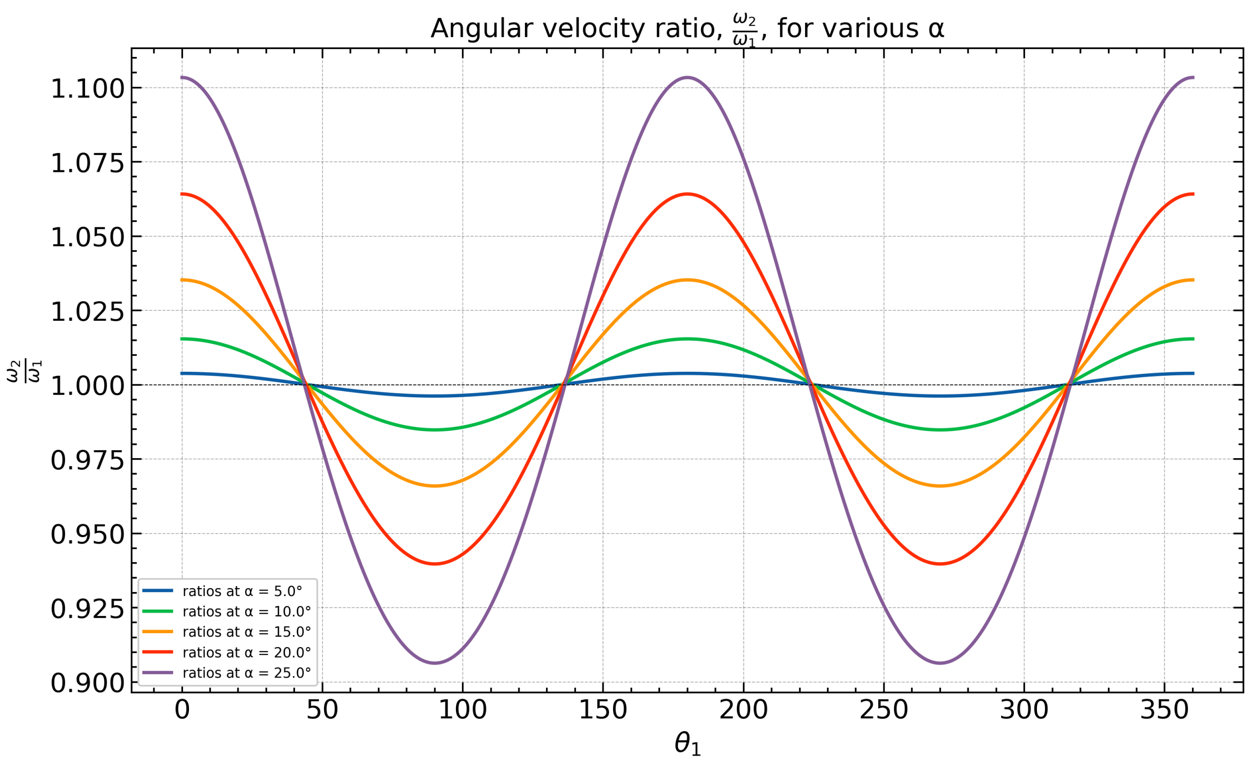

Figure 7: *Plot of $\frac{\omega_2}{\omega_1}$ as a function of θ_1 for several values of α.*

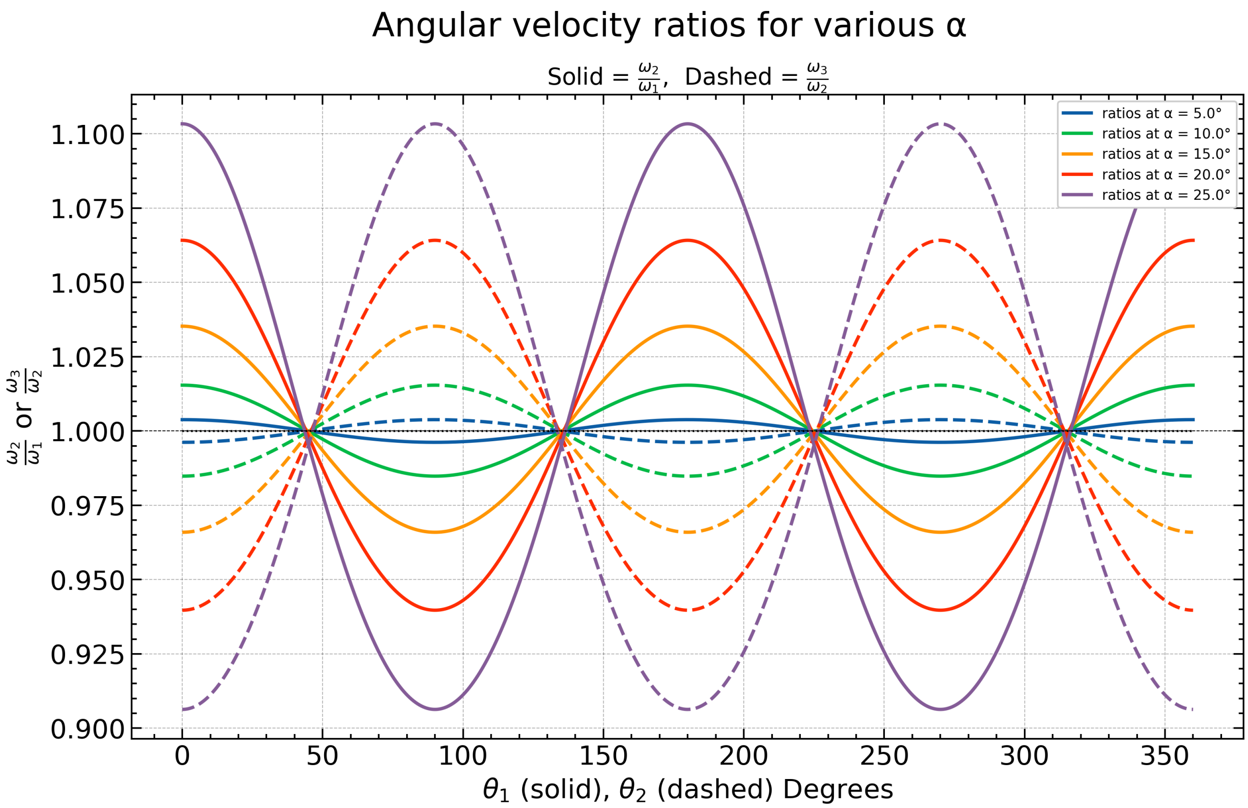

Figure 8: *Plot of $\frac{\omega_2}{\omega_1}$ (solid line) and $\frac{\omega_3}{\omega_2}$ (dashed line). The ratios are inverse of each other and $\frac{\omega_3}{\omega_1}$ is a horizontal dashed line at 1 - the cylic velocity has indeed canceled.*

We can make the following observations:

- The velocity cyle amplitude increaes as α increases.
- The velocity ratios are at unity for $\frac{(2n+1)\pi}{4}$, $(45^o, 135^o$, and so on) where $\sin(\theta) = \cos(\theta)$.
- The angled axis-
  - Rotates fastest when its yoke aligns with the intersection of the two rotating planes (0 and π in our frame) and is traversing the major axis of the ellipse.
  - Rotates slowest when its yoke is perpendicular to the intersection of the two rotating planes $(\frac{\pi}{2}, and \frac{3\pi}{2}$ in our frame) and is traversing the minor axis of the ellipse.
- The velocity ratios for the two joints are inversions of each other.

Figure 8 also shows $\frac{\omega_3}{\omega_1}$, the ratio of transfer case shaft velocity to differential pinion shaft velocity. The dashed line straight across the plot and always equal to 1 is a bit hard to see, but the cyclic velocity introduced at Joint 1 has indeed canceled across Joint 2. Well, almost…

In the real world, it is difficult to make the pinion shaft exactly parallel to the transfer case shaft. Morever, the pinion angle varies as the suspension moves up and down to accommodate uneven driving surfaces. So, surprisingly to many, there is always some non-uniform velocity at the differential pinion shaft.

If the offset from ideal is small the drive won't notice. As the offset grows the driver will experience vibrations - often more pronounced at certain speeds where shaft vibrations harmonize with suspension or steering system characteristics. It isn't uncommon to feel these in lifted vehicles where the suspension has been modified significantly from stock, but the proper pinion angle has not been re-established. Over time, an improper setup can damage driveline and steering components.

Figure 9 plots $\frac{\omega_3}{\omega_1}$ (pinion shaft over trasfer case shaft velocity) for a few different values of α where there is an offset between the angles at joint 1 and 2.

The practical implication: when a vehicle suspension is lifted, α increases because the vertical distance from transfer case to differential increases. To avoid driveline vibration, it is even more critical to minimize the offset between α_1 and α_2. The higher your ride, the more accurately you need to set the pinion angle.

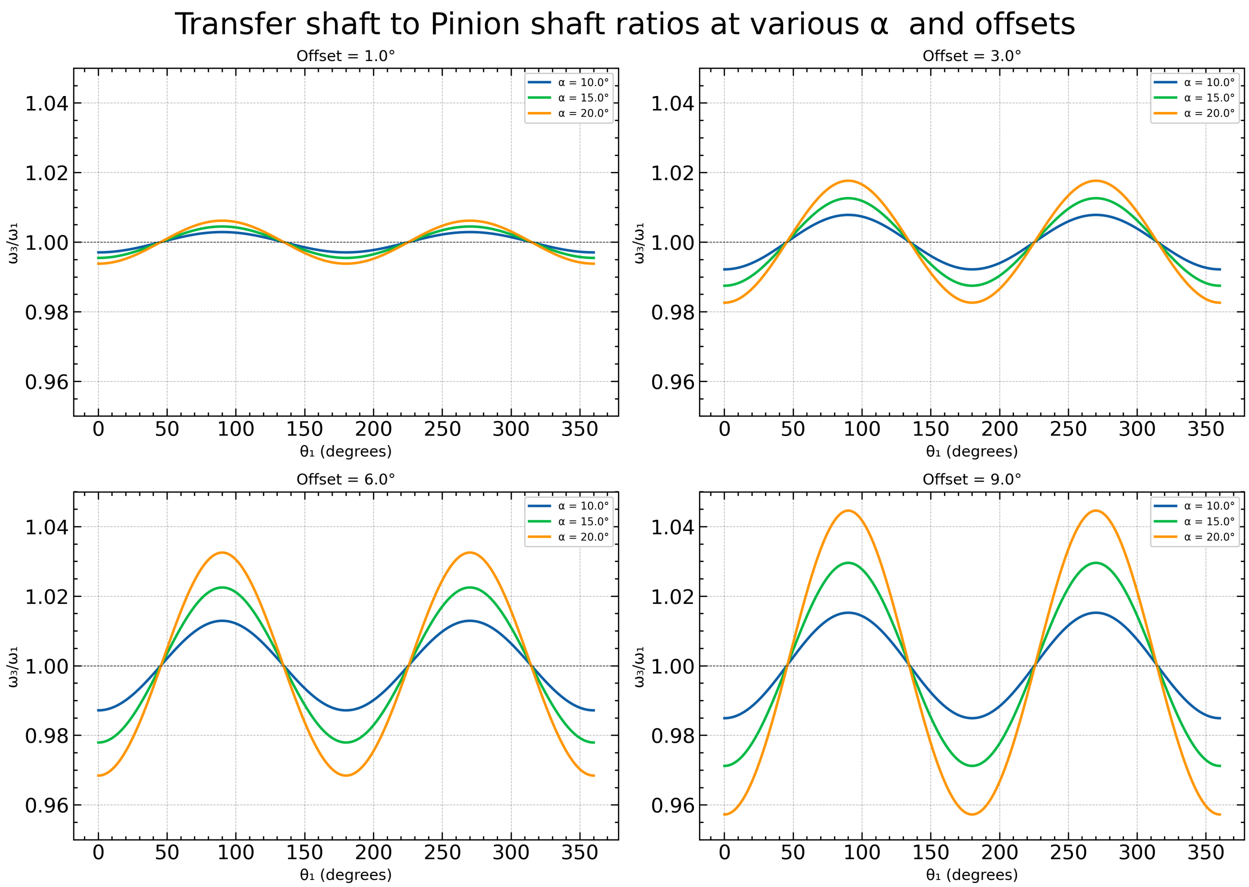

Figure 9: *Plots of $\frac{\omega_3}{\omega_1}$ (transfer case to differential pinion shaft velocity ratio) for several alphas and several offsets between α_1 and α_2. As α increases it becomes increasingly critical to minimize any offset.*

Figure 10 plots the same ratios as in Figure 7, but this time we imagine an inexperienced machine shop has bult our propeller shaft so that the yokes are perpendicular. We can see how the non-uniformity is compounded - not canceled.

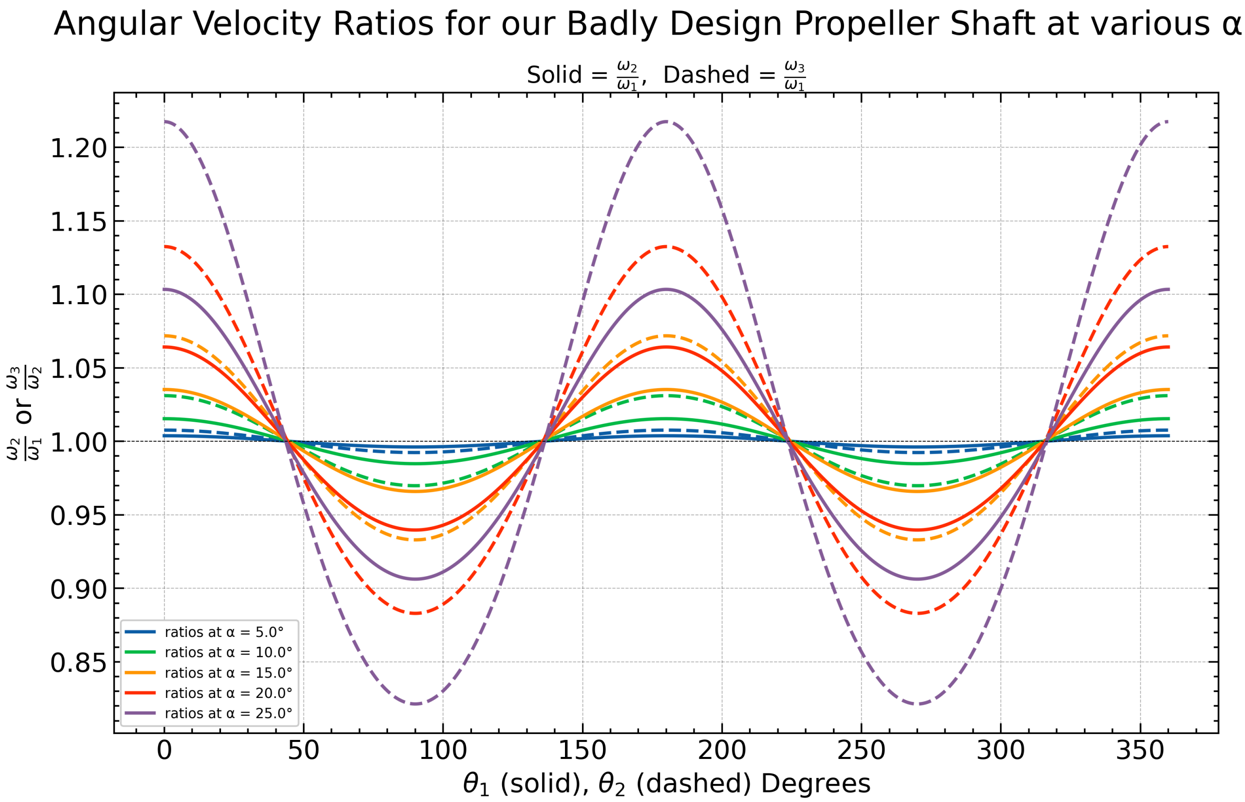

Figure 10: *Plot of* $\frac{\omega_3}{\omega_1}$ *An inexperienced machine shop has set our propeller shaft yokes perpendicuar and the cyclic velocity problem is compounded, not canceled.*

Make that a Double

At the start of this article, I told that my propeller shaft, with a Cardan joint at each end, required the differential pinion shaft to be parallel to the transfer case ouput shaft (and we showed why in the math). But not all propeller shafts are the same...

Figure 11 shows a different kind of propeller shaft that has a Cardan joint at one end and back-to-back cardan joints at the other end - three Cardan joints in total. My design is called a 'single Cardan' shaft and this design is called 'double Cardan' shaft. (Yes, two Cardans is a single and three Cardans is a double - I don't make the rules.)

Tom Woods Custom Double Cardan Jeep Drive Shaft

Figure 11: *A Tom Woods Custom Jeep Drive Shaft. A single Cardan joint at the differential and at the transfer case a double Cardan joint - back to back Cardans with a short section of shaft in between. For this design, the pinion should point straight at the transfer case to keep the angle at the single as close to zero as possible.*

The astute reader may already intuit that, for the double Cardan shaft, the differential pinion needs to point directly at the transfer case. Why? Well, the double cardan joint is at the transfer case side and the back-to-back joints will split the angle there and cyclic velocities will cancel. But, there is no fourth joint to cancel cyclic velocities at the differential side. Instead, that angle is made zero (or very close) by pointing the pinion at the transfer case - the entire angle required by the elevation difference is taken up at the back-to-back Cardan joint. Kind of…

In the real world, we need the single cardan joint because the suspension moves. But suspension movements is partly taken up at the double joint so the offset should be small and transient. For big suspension lifts, the double cardan is preferred and, if you look under a big lifted Jeep, you will likely see the differential pinion pointing right at the transfer case.

My Jeep has a double Cardan front propeller shaft, but not because of massive lift (I run a modest lift). It turns out that the front pinion angle is constrained by a steering alignment parameter called caster, and the double Cardan shaft does a better job of accommodating some inherent offset. Jeep designed the front so that the double joint eats most of the angle. There is more to the story, but we need to save that for another day as we will have to pull in steering configuration and alignment geometries that I don't fully understand yet.

In any case, Figure 12 illustrates the proper set up for a double cardan drive shaft.

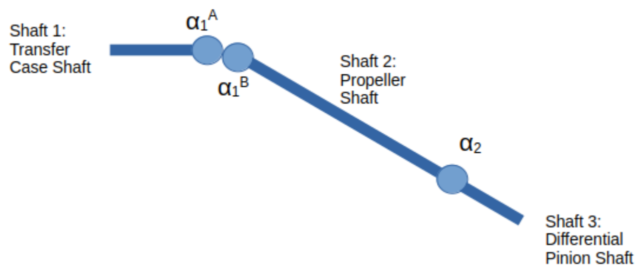

Figure 12: *The proper configuration for a double cardan drive shaft. Here the differential pinion is pointed straight at the transfer case to minimize any angle at the single joint.*

And now you know the dark math. Python code is attached as an Appendix.

Appendix – Python Code

```
#==============================================================================================
#==============================================================================================
#-------- Program to find angular velocity ratios across individual and combined Cardan joints
#==============================================================================================
#**********************************************************************************************


#===================================
#-import all the necessary stuff
#=================================

import numpy as np
import matplotlib.pyplot as plt
import sympy as sp
from matplotlib.animation import FuncAnimation
from matplotlib.animation import FuncAnimation, PillowWriter
import sys
import scienceplots



#===================================
# establish the math describing ratios across joints
#=================================


theta_v = np.linspace(0, 2*np.pi, 5000)
alphav = np.radians([5, 10, 15, 20, 25, 30])
alphav_short = np.radians([5, 10, 15])
alpha_offs = np.radians([1, 2, 4, 7])
w2_w1_v = lambda Theta1, Alpha1:  np.cos(Alpha1) / (1 - (np.cos(Theta1)**2)*(np.sin(Alpha1)**2
w3_w2_v = lambda Theta2, Alpha2:  np.cos(Alpha2) / (1 - (np.sin(Theta2)**2)*(np.sin(Alpha2)**2
theta2_calc = lambda Theta1, Alpha1: np.atan(np.tan(Theta1) / np.cos(Alpha1))
w3_w1_v = lambda Theta1, Alpha1, Alpha2:  w3_w2_v(theta2_calc(Theta1, Alpha1), Alpha2) * w2_w1


#=================================================================
# plot 1:  velocity ratios individual joints
#=================================================================
plt.figure(figsize=(12, 7))
for Alpha_v in alphav:
    ratio_21 = w2_w1_v(theta_v, Alpha_v)
    ratio_32 = w3_w2_v(theta_v, Alpha_v)
    ratio_31 = w3_w1_v(theta_v, Alpha_v, Alpha_v)
```

```
  # Plot w1/w2 solid and capture the color
    line1 = plt.plot(np.degrees(theta_v), ratio_21,
                     label=f'w2/w3 at alpha = {np.degrees(Alpha_v):.1f}',
                     linewidth=2)

    # Plot w2/w3 dashed with the same color
    plt.plot(np.degrees(theta_v), ratio_32,
             label=f'w2/w3 at alpha = {np.degrees(Alpha_v):.1f}',
             linewidth=2,
             linestyle='--',
             color=line1[0].get_color())

#   plot w3/w1 is a dotted  black line
    plt.plot(np.degrees(theta_v), ratio_31, linewidth=1, linestyle = ':',  color='k')

#  plot characteristics
plt.xlabel('theta_1 (solid), theta_2 (dashed) Degrees', fontsize=12)
plt.ylabel('w2/w1, or w3/w2', fontsize=12)
plt.suptitle('Driven to Drive shaft angular velocity ratio for various alpha', fontsize=14)
plt.title('Solid = w2/w1,  Dashed = w3/w2')
plt.legend(fontsize='small')
plt.grid(True, alpha=0.3)
plt.axhline(y=1, color='k', linewidth=0.5, linestyle='--')

# For plot 1 (single plot)
plt.savefig('single_joint_ratios.png', dpi=300, bbox_inches='tight')
plt.show()

#=======================================================================
#  plot #2:  velocities across combined joints with small offsets
#====================================================================
fig, axes = plt.subplots(2, 2, figsize=(14, 10))
axes = axes.flatten()

for idx, alpha_off in enumerate(alpha_offs):
    ax = axes[idx]

    for Alpha_vs in alphav_short:
        ratio_31 = w3_w1_v(theta_v, Alpha_vs - alpha_off, Alpha_vs)
        ax.plot(np.degrees(theta_v), ratio_31,
                label=f'alpha = {np.degrees(Alpha_vs):.1f}',
                linewidth=2)

    ax.set_xlabel('theta_1 (degrees)', fontsize=12)
    ax.set_ylabel('w3/s1', fontsize=12)
    ax.set_title(f'Offset = {np.degrees(alpha_off):.1f}', fontsize=12)
    ax.legend(fontsize='small')
```

```python
    ax.grid(True, alpha=0.3)
    ax.axhline(y=1, color='k', linewidth=0.5, linestyle='--')

plt.suptitle('Drive to Driven shaft angular velocity ratio for various alpha offsets', fontsize
plt.tight_layout()


plt.savefig('overall_ratio.png', dpi=300, bbox_inches='tight')
plt.show()


#===================================================================
# Plot 3:  combined ratios with propeller shaft ears perpendicular
#===================================================================
plt.figure(figsize=(12, 7))
for Alpha_v in alphav:
    ratio_21 = w2_w1_v(theta_v, Alpha_v)
    ratio_31b = ratio_21**2

  # Plot w1/w2 solid and capture the color
    line1 = plt.plot(np.degrees(theta_v), ratio_21,
                     label=f'w1/w2 at alpha = {np.degrees(Alpha_v):.1f}', linewidth=2)

    # Plot w3/w1 dashed with the same color
    plt.plot(np.degrees(theta_v), ratio_31b,
             label=f'w3/w1 at alpha = {np.degrees(Alpha_v):.1f}',
             linewidth=2,
             linestyle='--',
             color=line1[0].get_color())



#  plot characteristics
plt.xlabel('theta_1 (solid), theta_2 (dashed) Degrees', fontsize=12)
plt.ylabel('w2/s1 or w3/w1', fontsize=12)
plt.suptitle('Angular Velocity Ratios for our Badly Design Propeller Shaft at various alpha', f
plt.title('Solid = w2/w1,  Dashed = w3/w1')
plt.legend(fontsize='small')
plt.grid(True, alpha=0.3)
plt.axhline(y=1, color='k', linewidth=0.5, linestyle='--')

# For plot 1 (single plot)
plt.savefig('badshaft_joint_ratios.png', dpi=300, bbox_inches='tight')
plt.show()
```

```
#===================================================
#  END  END  END  END
```